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# The geometric interpretation of inversion formulae for rational plane curves

Wenping Wang [a], Barry Joe [b,*]

[a] Department of Computer Science, University of Hong Kong, Pokfulam Road, Hong Kong
[b] Department of Computing Science, University of Alberta, Edmonton, AB, Canada T6G 2H1

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**Abstract**

Given a faithful parameterization $P(t)$ of a rational plane curve, an inversion formula $t = f(x,y)$ gives the parameter value corresponding to a point $(x,y)$ on the curve, where $f$ is a rational function in $x$ and $y$. We investigate the relationship between a point $(x^*, y^*)$ not on the curve and the corresponding point $P(t^*)$ on the curve, where $t^* = f(x^*, y^*)$. It is shown that for a rational quadratic plane curve, $P(t^*)$ is the projection of $(x^*, y^*)$ from a point which may be any point on the curve; for a rational cubic plane curve, $P(t^*)$ is the projection of $(x^*, y^*)$ from the double point of the curve. Applications of these results are discussed and a generalized result is proved for rational plane curves of higher degree.

## 1. Introduction

Rational curves are widely used in CAGD. Here we study one problem in the application of rational plane curves. Given a point $P = (x, y)$ on a faithful rational plane curve $P(t)$, sometimes it is necessary to know the corresponding parameter value. A formula that gives this parameter value $t$ is called an inversion formula of $P(t)$. But given a point $P^* = (x^*, y^*)$ not on the curve, a parameter value $t^*$ can also be obtained through the inversion formula, in general. Here we do not concern ourselves with finding the inversion formulae of a rational curve, for which the reader is referred to (Goldman et al., 1984). This paper focuses on revealing the geometric relationship between $P^*$ and the corresponding point $P(t^*)$ on the curve. This question is posed and briefly addressed in (Goldman et al., 1984).

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In order to clearly formulate the problem and present the solution, we first make some conventions on notation. $\mathcal{C}$ denotes the field of complex numbers, and $\mathcal{R}$ its subfield of reals. For a field $K$, $K[t]$ and $K[x,y]$ denote the domains of polynomials in one and two variables over $K$, respectively; and $K(t)$ and $K(x,y)$ denote the quotient fields of $K[t]$ and $K[x,y]$, respectively. For instance, $\mathcal{R}(t)$ is the field of rational functions of $t$ with coefficients in $\mathcal{R}$. In the following discussion all algebraic equations are assumed to have coefficients in $\mathcal{R}$. A complex point $A = (a_1, a_2)$ in the affine plane $\mathcal{C}^2$ is called a real point if $a_1$ and $a_2 \in \mathcal{R}$. $\bar{A} = (\bar{a}_1, \bar{a}_2)$ denotes the conjugate point of $A$, where $\bar{a}_i$ is the conjugate of $a_i$, $i = 1, 2$. Note that if $A$ satisfies a polynomial equation or a system of polynomial equations with real coefficients, so does $\bar{A}$.

We will consider the rational plane curve $U$ given by the parameterization

$$P(t) = (x(t), y(t)) = (f_1(t)/g(t), f_2(t)/g(t)), \tag{1}$$

where $f_1(t)$, $f_2(t)$ and $g(t) \in \mathcal{R}[t]$ have no common factor. The degree of $P(t)$ is denoted by $\deg(P(t)) = \max\{\deg(f_1), \deg(f_2), \deg(g)\}$. In CAGD applications it is usually assumed that $t \in \mathcal{R}$ so that (1) yields a real locus. But we shall often consider the case in which $t \in \mathcal{C}$, mainly because the algebraic closure of $\mathcal{C}$ facilitates our theoretical development. Nevertheless, we will indicate from time to time whether the obtained results are valid for the case of $t \in \mathcal{R}$. Curves resulting from the two cases are denoted by $U(\mathcal{R})$ and $U(\mathcal{C})$, respectively; and obviously, $U(\mathcal{R}) \subset U(\mathcal{C})$. When we do not wish to distinguish these two cases, the curve is simply referred to as $U$. In order to carry out our discussion in a complete sense, we will sometimes resort to points at infinity and use homogeneous coordinates, and the parameter of a curve will be allowed to take the improper value $\infty$. Accordingly, $U(\mathcal{R})$ and $U(\mathcal{C})$ are assumed to contain all real and complex points of the curve at infinity, respectively. $P(t)$ is called a parameterization of the curve $U$, which is treated as the locus traced out by $P(t)$ as $t$ runs through $\mathcal{R}$ or $\mathcal{C}$. Evidently no two distinct points on $U(\mathcal{C})$ correspond to the same parameter value. We also call $P(t)$ a curve when there is no danger of confusion.

**Definition.** $P(t)$ is called a *faithful parameterization* of $U(\mathcal{C})$ if, with at most a finite number of exceptions, distinct values of $t \in \mathcal{C}$ give distinct points on $U(\mathcal{C})$.

Replacing $\mathcal{C}$ by $\mathcal{R}$ in the above definition, we can define a faithful parameterization of a real curve $U(\mathcal{R})$. It is evident that $P(t)$ is a faithful parameterization of $U(\mathcal{R})$ if it is a faithful parameterization of $U(\mathcal{C})$. We will see later that the converse is also true. By the Lüroth Theorem (Walker, 1950) any rational curve has a faithful parameterization. Also a parameterization is faithful if and only if it is of the lowest possible degree among all the parameterizations of $U$ (Sederberg, 1986). Hence from now on, we assume that any $P(t)$ we discuss is a faithful parameterization of $U$. With this assumption, it is sensible to call $U$ a rational plane curve of degree $n = \deg(P(t))$, which is independent of any particular parameterization of $U$.

Given a point $(x,y)$ on a rational curve $U(\mathcal{R}) : P(t)$, there is a formula $t = f(x,y) \in \mathcal{R}(x,y)$ giving the corresponding parameter value $t \in \mathcal{R}$, with only a finite number of exceptional points $(x,y)$. This is also true for a point $(x,y)$ on $U(\mathcal{C})$, but with the resulting $t \in \mathcal{C}$. Such a formula is called an inversion formula of $P(t)$. It is known that an inversion formula of a rational curve of degree $n \geqslant 3$ can be expressed as $t = N(x,y)/M(x,y)$, where $N(x,y)$ and $M(x,y) \in \mathcal{R}[x,y]$ are of degree at most $n-2$; and there exist curves for which the upper bound $n-2$ can not be reduced. For a rational quadratic curve an inversion formula is $t = N(x,y)/M(x,y)$, where $N(x,y)$ and $M(x,y) \in \mathcal{R}[x,y]$ are linear in $x$ and $y$. These results about inversion formulae can be found in (Goldman et al., 1984).

If a rational curve $P(t)$ is faithful in $\mathcal{R}$, then there exists an inversion formula $t = N(x,y)/M(x,y)$ of $P(t)$. It follows that $P(t)$ is also faithful in $\mathcal{C}$ since this inversion formula is also valid for points on $U(\mathcal{C})$. The only possible exceptional points on the curve that correspond to more than one parameter values are intersections of $M(x,y) = 0$ and $N(x,y) = 0$, therefore their number is finite.

Given an inversion formula $t = f(x,y)$ of $P(t)$, a point $(x^*, y^*)$ not on the curve also corresponds to a parameter value if substituted into $t = f(x,y)$, therefore there is the question of what is the relationship between the point $(x^*, y^*)$ and the corresponding point $P(t^*)$ on the curve, where $t^* = f(x^*, y^*)$. By the rational linearity of $f(x,y)$ in the case of rational quadratic and cubic curves, it immediately follows that $t = f(x,y)$ determines a family of straight lines with one parameter $t$, which is easily seen to be a pencil of lines. Thus a preliminary interpretation of the inversion formulae for rational quadratic and cubic curves is that $P(t^*)$ is the projection of $P(x^*, y^*)$ from some point $A$, which is the center of the pencil defined by $t = f(x,y)$ (Goldman et al., 1984).

In the following sections we will study in detail the inversion formulae in the case of rational curves of degree $n = 2$ and 3. The main results are that in the case of $n = 2$, $P(t^*)$ is the projection of $(x^*, y^*)$ from a point $A$ which can be any point on the curve, with different points corresponding to different inversion formulae of the same parameterization $P(t)$; in the case of $n = 3$, $P(t^*)$ is always the projection of $(x^*, y^*)$ from the only double point of the curve. We will also generalize the above result for rational plane curves of degree $n \geqslant 4$, that is, if $t = N(x,y)/M(x,y)$, $N(x,y)$ and $M(x,y) \in \mathcal{R}[x,y]$, is an inversion formula of a rational plane curve $P(t)$, then $N(x,y) = 0$ and $M(x,y) = 0$ both pass through all the singular points of $P(t)$.

The remainder of this paper is organized in the following way. In Section 2 some preliminaries are reviewed; they are mainly basic and relevant results in the theory of algebraic curves. In Section 3 we explain the interpretation of inversion formulae for rational linear and quadratic curves. Rational cubic curves are addressed separately in Section 4 due to their importance in CAGD applications. Some general results are discussed in Section 5.

## 2. Preliminaries

This section contains some facts about rational plane curves to be used later. A faithful rational plane curve $P(t)$ of degree $n$ is an irreducible algebraic curve of order $n$, with an algebraic equation being $F(x,y) = 0$, where $F(x,y) \in \mathcal{R}[x,y]$ is of degree $n$. $F(x,y) = 0$ is also called the implicit form or implicit equation of $P(t)$. More precisely, every point of $F(x,y) = 0$ corresponds to at least one parameter $t \in \mathcal{C}$; and every real point of $F(x,y) = 0$ corresponds to at least one parameter $t \in \mathcal{R}$, except isolated real points of $F(x,y) = 0$, which correspond to some $t \in \mathcal{C}$, e.g. an acnode of a cubic. Furthermore, every point (or real point) of $F(x,y) = 0$ corresponds to a unique parameter value $t \in \mathcal{C}$ (or $t \in \mathcal{R}$), in the sense that it is yielded by $P(t)$, with only a finite number of exceptions. These exceptions are all ordinary singular points of $F(x,y) = 0$. Hence we can reasonably use $U(\mathcal{R})$ to denote the set of all real points of $F(x,y) = 0$, excluding all the isolated real points, while using $U(\mathcal{C})$ to denote curve $F(x,y) = 0$ in the complex affine plane $\mathcal{C}^2$.

A point $(x_0, y_0)$ is called a *k-fold point* of $F(x,y) = 0$ if all $j$-th partial derivatives of $F(x,y)$, $j = 0, 1, \ldots, k-1$, vanish at $(x_0, y_0)$, and at least one $k$-th partial derivative of $F(x,y)$ does not vanish at $(x_0, y_0)$. $k$ is called the *multiplicity* of $(x_0, y_0)$ on the curve $F(x,y) = 0$. A point not on the curve is said to have multiplicity 0. A $k$-fold point is said to be *singular* if $k \geqslant 2$, while a singular point with $k = 2$ is also called a *double point.* A singular point with all its nodal tangents being distinct is called an *ordinary singular point*, otherwise a *nonordinary singular point.* An algebraic curve is said to be rational if it has a rational parameterization. It is well known that an irreducible algebraic curve of order $n$ is rational if and only if it contains $\frac{1}{2}(n-1)(n-2)$ double points (Walker, 1950), assuming that the number of double points in a multiple point is counted properly, i.e. a $k$-fold point accounts for at least $\frac{1}{2}k(k-1)$ double points. $\Delta = \frac{1}{2}(n-1)(n-2)$ is the maximum number of double points an irreducible algebraic curve of order $n$ can possess.

From the above facts it follows that a nondegenerate rational quadratic curve has no singular point. A rational cubic plane curve has exactly one double point, which may be a crunode, a cusp, or an acnode (Patterson, 1988). This only double point, finite or not, must be real, for if the double point $Q$ is not real, then its conjugate point $\bar{Q} \neq Q$ will also be a double point of $P(t)$, contradicting the fact that an irreducible cubic can have at most one double point. The configuration of singular points of a higher degree curve can be very involved. For this the interested reader may consult books on algebraic curves, e.g. (Walker, 1950).

Given the rational parameterization of a curve, its implicit form, or the algebraic equation, can be found using resultants (Goldman et al., 1984). Here we are more interested in the converse problem, i.e. finding a parameterization of a rational curve from its implicit form. It is important to note that both problems have closed form solutions with respect to the coefficient field; for instance, if the algebraic equation is given by $F(x,y) = 0$ with $F(x,y) \in \mathcal{R}[x,y]$, then the resulting parameterization should be $P(t) = (x(t), y(t))$ with $x(t)$ and $y(t) \in \mathcal{R}(t)$. Here we will briefly review how to parameterize algebraic curves of order $n = 2$ or 3. Given an algebraic curve $F(x,y) = 0$ of order $n = 2$ or 3, which is rational, where $F(x,y) \in \mathcal{R}[x,y]$, select a point $A = (a_1, a_2) \in U(\mathcal{C})$ , which may be any point on the curve when $n = 2$, but must be the (real) double point when $n = 3$. By this choice of $A$, the straight line $y - a_2 = t(x - a_1)$, $t \in \mathcal{C}$, intersects the curve at exactly one other point $P(t) = (x(t), y(t))$ besides $A$, which yields a rational parameterization of $F(x,y) = 0$. For $A \in U(\mathcal{R})$ it can be shown that $x(t)$ and $y(t) \in \mathcal{R}(t)$; otherwise, $x(t)$ and $y(t) \in \mathcal{C}(t)$. A comprehensive discussion of parameterizing algebraic curves of arbitrary order that are rational can be found in (Abhyankar et al., 1988).

The following lemma is needed later. Since it is a basic result about algebraic curves (Walker, 1950), it is given without proof.

**Lemma 1.** *Let U be an algebraic curve of order n without multiple components in the complex projective plane. Through an r-fold point A of U,* $0 \leqslant r \leqslant n$, *there exists a straight line that intersects the curve U in another* $n-r$ *distinct points besides A.*

## 3. Interpretation for rational linear and quadratic curves

In this section we first use the rational linear curves as an example to illustrate some basic problems in the study of inversion formulae.

Let a straight line be represented as the rational linear curve $P(t)=(x(t),y(t))$, where

$$x(t) = \frac{a_0 t + a_1}{c_0 t + c_1}, \qquad y(t) = \frac{b_0 t + b_1}{c_0 t + c_1}, \tag{2}$$

and $(b_0c_1 - b_1c_0)^2 + (c_0a_1 - c_1a_0)^2 \neq 0$. This line has implicit equation

$$(b_0c_1 - b_1c_0)x + (c_0a_1 - c_1a_0)y + a_0b_1 - a_1b_0 = 0.$$

From (2) we immediately obtain two inversion formulae for $P(t)$,

$$t = -\frac{a_1 - c_1 x}{a_0 - c_0 x} \quad \text{and} \quad t = -\frac{b_1 - c_1 y}{b_0 - c_0 y},$$

in each case assuming that the denominator does not vanish identically. It is easy to tell the interpretation of these inversion formulae. For example, all points having the same abscissa yield the same parameter value through the first formula, i.e. given point $(x^*, y^*)$ not on the straight line, its corresponding point $P(t^*)$, $t^* = -(a_1 - c_1x^*)/(a_0 - c_0x^*)$, is the projection of $(x^*, y^*)$ parallel to the $y$ axis. Notice that the inversion formula for a rational linear curve is not unique. In general, we want to know whether the inversion formula for a rational plane curve is unique, as well as the geometric interpretation of an inversion formula. In the case of the rational linear curve, all its rational linear inversion formulae are characterized by the following theorem.

**Theorem 1.** *Given any point with homogeneous coordinates* $(\tilde{X}, \tilde{Y}, \tilde{W})$ *not on the line* (2), *there is an inversion formula for* (2) *of the form*

$$t = f(x, y) \equiv \frac{N(x, y)}{M(x, y)},$$

*where* $N(x,y) = 0$ *and* $M(x,y) = 0$ *are two straight lines intersecting at* $(\tilde{X}, \tilde{Y}, \tilde{W})$.

**Proof.** Let the homogeneous expression of (2) be $(X(t), Y(t), W(t))$, where $x(t) = X(t)/W(t)$, $y(t) = Y(t)/W(t)$ and $X(t)$, $Y(t)$, and $W(t)$ are all linear in $t$. Let

$$\tilde{M}(x,y) = \begin{vmatrix} x & \tilde{X} & X(\infty) \\ y & \tilde{Y} & Y(\infty) \\ 1 & \tilde{W} & W(\infty) \end{vmatrix}, \qquad \tilde{N}(x,y) = \begin{vmatrix} x & \tilde{X} & X(0) \\ y & \tilde{Y} & Y(0) \\ 1 & \tilde{W} & W(0) \end{vmatrix}. \tag{3}$$

$\tilde{M}(x,y) = 0$ and $\tilde{N}(x,y) = 0$ are the equations of the lines through $(\tilde{X},\tilde{Y},\tilde{W})$, $(X(\infty),Y(\infty),Z(\infty))$ and $(\tilde{X},\tilde{Y},\tilde{W})$, $(X(0),Y(0),Z(0))$, respectively, so the lines $\tilde{M}(x,y) = 0$ and $\tilde{N}(x,y) = 0$ intersect at $(\tilde{X},\tilde{Y},\tilde{W})$. Note also that $\tilde{f}(x,y) \equiv \tilde{N}(x,y)/\tilde{M}(x,y)$ is rational linear in $x$ and $y$; therefore $\tilde{f}(x(t),y(t))$ is rational linear in $t$. Let

$$\tilde{f}(x(t),y(t)) = \frac{\lambda_1 t + \mu_1}{\lambda_2 t + \mu_2}.$$

By (3) $\tilde{f}(x(0),y(0)) = 0$, $\tilde{f}(x(\infty),y(\infty)) = \infty$. Therefore $\mu_1 = \lambda_2 = 0$ and $\lambda_1 \neq 0$, $\mu_2 \neq 0$. Hence

$$\tilde{f}(x(t),y(t)) = \frac{\lambda_1 t}{\mu_2}.$$

The proof is completed by letting $f(x,y) = (\mu_2/\lambda_1)\tilde{f}(x,y)$. □

We now turn to the study of rational quadratic curves. All faithful rational quadratic curves are nondegenerate conic sections. Given a rational quadratic curve $U$: $P(t) = (x(t),y(t))$,

$$x(t) = \frac{a_2t^2 + a_1t + a_0}{c_2t^2 + c_1t + c_0}, \qquad y(t) = \frac{b_2t^2 + b_1t + b_0}{c_2t^2 + c_1t + c_0}, \tag{4}$$

its inversion formula can be obtained by solving for $t$ from the system

$$\begin{aligned} &(c_2x - a_2)t^2 + (c_1x - a_1)t + (c_0x - a_0) = 0, \\ &(c_2y - b_2)t^2 + (c_1y - b_1)t + (c_0y - b_0) = 0. \end{aligned} \tag{5}$$

The solution is

$$t = \frac{-[(b_2c_0 - b_0c_2)x + (a_0c_2 - a_2c_0)y + (a_2b_0 - a_0b_2)]}{(b_2c_1 - b_1c_2)x + (a_1c_2 - a_2c_1)y + (a_2b_1 - a_1b_2)} \equiv \frac{N(x,y)}{M(x,y)}, \tag{6}$$

where $M(x,y)$ and $N(x,y) \in \mathcal{R}[x,y]$. Eq. (6) represents a family of straight lines with one parameter $t$. Its envelope is the intersection of $M(x,y) = 0$ and $N(x,y) = 0$, which will be called the *center* of the inversion formula, which is rational linear. The intersection of $M(x,y) = 0$ and $N(x,y) = 0$ is taken to be a point at infinity in case they are parallel to each other. Since $t = N(x,y)/M(x,y)$ is not a constant, straight lines $M(x,y) = 0$ and $N(x,y) = 0$ are not coincident.

**Theorem 2.** *The center $A$ of the inversion formula* (6) *is on the conic section defined by* (4). *And corresponding to any point $\tilde{A} \in U(\mathcal{R})$ there is an inversion formula $t = \tilde{N}(x,y)/\tilde{M}(x,y)$, with center at $\tilde{A}$, where $\tilde{N}(x,y)$ and $\tilde{M}(x,y) \in \mathcal{R}[x,y]$ are linear.*

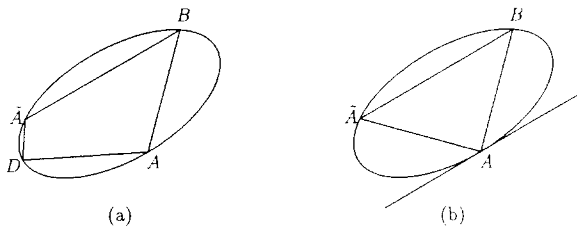


Fig. 1. Illustrations for the proof of Theorem 2.

**Proof.** First we remark that all points on the same line of the pencil of (6), except the center of the pencil, yield the same parameter value $t$ if substituted into (6).

Since $A$, as the intersection of $M(x,y) = 0$ and $N(x,y) = 0$, is real, $A \in U(\mathcal{C})$ implies $A \in U(\mathcal{R})$. Hence we suppose that $A$ is not on the curve $U(\mathcal{C})$. Then by Lemma 1 there exists a line $t_0M(x,y) - N(x,y) = 0$ that intersects $U(\mathcal{C})$ in exactly two distinct points $P_1$ and $P_2$. So the two points $P_1$ and $P_2$ have the same parameter value $t_0 \in \mathcal{C}$. This contradicts that $U(C)$ has parameterization $P(t)$.

Now we prove the second part of the theorem. First assume that neither $M(x,y) = 0$ nor $N(x,y) = 0$ is tangent to $U$. Let $\tilde{A}$ be any point on $U(\mathcal{R})$. Let $B$ and $D$ be the intersections of $M(x,y) = 0$ and $N(x,y) = 0$ with the curve $U(\mathcal{C})$ besides $A$, respectively. See Fig. 1(a). Apparently $B$ and $D \in U(\mathcal{R})$, for otherwise the real line $M(x,y) = 0$ or $N(x,y) = 0$ would intersect the conic in their conjugate points $\bar{B}$ and $\bar{D}$, respectively, which is impossible since they already pass through $A \in U(\mathcal{R})$. Let $l_{XY} = 0$ denote an equation of the straight line passing through points $X$ and $Y$, with $XY$ standing for $\tilde{A}B$, $AD$, $\tilde{A}D$ or $AB$. Then $l_{XY} \in \mathcal{R}[x,y]$. We assume $B \neq \tilde{A}$ and $D \neq \tilde{A}$; the case of $B = \tilde{A}$ or $D = \tilde{A}$ is similar to that below, where one of $M(x,y) = 0$ and $N(x,y) = 0$ is tangent to $U$ at $A$. By the familiar property of a conic pencil (Faux and Pratt, 1979), the equation of $U$ can be expressed as

$$l_{\tilde{A}B}l_{AD} - \lambda l_{\tilde{A}D}l_{AB} = 0, \tag{7}$$

where $\lambda$ is easily seen to be a real constant. In particular, we may choose $l_{AB} = M(x,y)$ and $l_{AD} = N(x,y)$. For any point $(x,y)$ on $U$, when $M(x,y) \neq 0$ and $l_{\tilde{A}B}(x,y) \neq 0$, we obtain, by (7)

$$t = N(x,y)/M(x,y) = \lambda \frac{l_{\tilde{A}D}}{l_{\tilde{A}B}}. \tag{8}$$

So $t = \lambda l_{\tilde{A}D}/l_{\tilde{A}B}$ represents a pencil with center at $\tilde{A}$. Denoting $\lambda l_{\tilde{A}D}$ and $l_{\tilde{A}B}$ by $\tilde{N}(x,y)$ and $\tilde{M}(x,y)$ respectively, we have $\tilde{M}(x,y)$ and $\tilde{N}(x,y) \in \mathcal{R}[x,y]$.

If one of $N(x,y) = 0$ and $M(x,y) = 0$ is tangent to $U$ at $A$, then the other is not, since they are not coincident. Without loss of generality assume that $N(x,y)$ is tangent to $U$. Then it can be shown that the equation of $U$ is still expressible by (7), but with $D$ being replaced by $A$ and $l_{AA}$ being interpreted as the tangent of $U$ through $A$. See Fig. 1(b). This also includes the case of either $B = \tilde{A}$ or $D = \tilde{A}$ mentioned above. The remainder of the proof is the same as the preceding case where $N(x,y) = 0$ and $M(x,y) = 0$ are not tangent to $U$. □

In (8) $t = 0$ implies $N(x,y) = 0$, therefore $D = P(0)$; similarly, $B = P(\infty)$. Thus for any $\tilde{A}$ the equations $l_{\tilde{A}D} = 0$ and $l_{\tilde{A}B} = 0$ can be easily obtained, and $\lambda$ can be obtained from the $x(t)$, $y(t)$, and $t$ value of any point on the curve which is not $\tilde{A}$, $B$ or $D$.

If a point $\tilde{A} \in U(\mathcal{C})$ is taken as the center of an inversion formula, we have the following theorem.

**Theorem 3.** *Given rational quadratic curve $U : P(t)$ defined by (4), and a point $\tilde{A} \in U(\mathcal{C})$, there corresponds a rational linear inversion formula $t = \tilde{N}(x,y)/\tilde{M}(x,y)$ of $P(t)$ with center at $\tilde{A}$, where $\tilde{M}(x,y)$ and $\tilde{N}(x,y) \in \mathcal{C}[x,y]$. Furthermore, $\tilde{M}(x,y)$ and $\tilde{N}(x,y) \in \mathcal{R}[x,y]$ if and only if $\tilde{A} \in U(\mathcal{R})$.*

**Proof.** The proof of the first part is similar to that of Theorem 2. We just need to prove the second part.

The sufficiency follows from Theorem 2. The necessity part is obvious by noting that if both $\tilde{M}(x,y)$ and $\tilde{N}(x,y) \in \mathcal{R}[x,y]$, then $\tilde{A} \in U(\mathcal{R})$, since it is the intersection of two real lines. □

The following theorem shows that a rational linear inversion formula with a given center is essentially unique.

**Theorem 4.** *Let $t = N(x,y)/M(x,y)$ be a rational linear inversion formula with center $A$ for a rational plane curve $U : P(t)$. If $t = \tilde{N}(x,y)/\tilde{M}(x,y)$ is another rational linear inversion formula of $P(t)$ with center $A$, then $\tilde{M}(x,y) = \lambda M(x,y)$ and $\tilde{N}(x,y) = \lambda N(x,y)$ for some nonzero constant $\lambda$.*

**Proof.** Since $\tilde{M}(x,y) = 0$ and $\tilde{N}(x,y) = 0$ each contain $A$, they belong to the pencil $N(x,y) - tM(x,y) = 0$. Therefore $\tilde{M}(x,y) = \lambda_1 M(x,y) + \mu_1 N(x,y)$ and $\tilde{N}(x,y) = \lambda_2 M(x,y) + \mu_2 N(x,y)$ for some constants $\lambda_1$, $\mu_1$, $\lambda_2$, $\mu_2$. So

$$t = \frac{\tilde{N}(x,y)}{\tilde{M}(x,y)} = \frac{\lambda_2 M(x,y) + \mu_2 N(x,y)}{\lambda_1 M(x,y) + \mu_1 N(x,y)}$$
$$= \frac{\lambda_2 + \mu_2 N(x,y)/M(x,y)}{\lambda_1 + \mu_1 N(x,y)/M(x,y)} = \frac{\lambda_2 + \mu_2 t}{\lambda_1 + \mu_1 t},$$

and it follows that $\lambda_2 = \mu_1 = 0$ and $\mu_2 = \lambda_1 \neq 0$. Thus $\tilde{M}(x,y) = \lambda_1 M(x,y)$ and $\tilde{N}(x,y) = \lambda_1 N(x,y)$. □

Now we consider some application issues of the inversion formulae of rational quadratic curves. We ask the following questions. Which point $P(t_0)$ is the center of the inversion formula given by (6)? How do inversion formulae with different centers arise naturally? If inversion formulae with different centers are available, which of them is preferable to the others?

Consider $P(t)$ defined by (4). Since $P(t)$ is faithful, given a point on the curve $U$, there is unique $t$ satisfying (5). Treating $t$ and $t^2$ as unknowns, $t$ can be solved for in two ways. The first expression is

$$t = -\frac{\begin{vmatrix} c_2x - a_2 & c_0x - a_0 \\ c_2y - b_2 & c_0y - b_0 \end{vmatrix}}{\begin{vmatrix} c_2x - a_2 & c_1x - a_1 \\ c_2y - b_2 & c_1y - b_1 \end{vmatrix}}, \tag{9}$$

which is the same as (6). Rewriting (5) as

$$(c_2x - a_2)t + (c_1x - a_1) + (c_0x - a_0)/t = 0,$$
$$(c_2y - b_2)t + (c_1y - b_1) + (c_0y - b_0)/t = 0,$$

and treating $t$ and $1/t$ as unknowns, we obtain another expression of $t$,

$$t = -\frac{\begin{vmatrix} c_1x - a_1 & c_0x - a_0 \\ c_1y - b_1 & c_0y - b_0 \end{vmatrix}}{\begin{vmatrix} c_2x - a_2 & c_0x - a_0 \\ c_2y - b_2 & c_0y - b_0 \end{vmatrix}}. \tag{10}$$

**Theorem 5.** *The inversion formula* (9) *has its center at* $P(\infty)$. *The inversion formula* (10) *has its center at* $P(0)$.

**Proof.** Consider (9) first. To facilitate the discussion of the intersection of two straight lines in the affine plane we will use homogeneous coordinates. Let $x = X/W$ and $y = Y/W$. The center of (9) is the intersection of two straight lines,

$$(b_2c_0 - b_0c_2)X + (a_0c_2 - a_2c_0)Y + (a_2b_0 - a_0b_2)W = 0$$

and

$$(b_2c_1 - b_1c_2)X + (a_1c_2 - a_2c_1)Y + (a_2b_1 - a_1b_2)W = 0,$$

which are, in fact, two straight lines passing through the pair of points $(a_0, b_0, c_0)$ and $(a_2, b_2, c_2)$, and the pair of points $(a_1, b_1, c_1)$ and $(a_2, b_2, c_2)$, respectively. Therefore they intersect at $(a_2, b_2, c_2)$, which is $P(\infty)$. Note that $(a_i, b_i, c_i)$, $i = 0, 1, 2$, are distinct points since $P(t)$ is nondegenerate.

A similar argument shows that the center of (10) is $P(0)$. □

From the above results it is clear that for rational quadratic curves different inversion formulae arise naturally when different approaches are employed to find them. In CAGD the Bézier representation of curves is very popular. So here we illustrate the above results by some examples in the setting of rational quadratic Bézier curves.

**Example 1.** Let $P(t)$ be a rational parameterization of the semicircle $x^2 + y^2 = 1, x \geqslant 0$, given by the Bézier curve

$$P(t) = \frac{P_0B_{0,2}(t) + P_1B_{1,2}(t) + P_2B_{2,2}(t)}{B_{0,2}(t) + B_{2,2}(t)}, \quad t \in [0, 1], \tag{11}$$

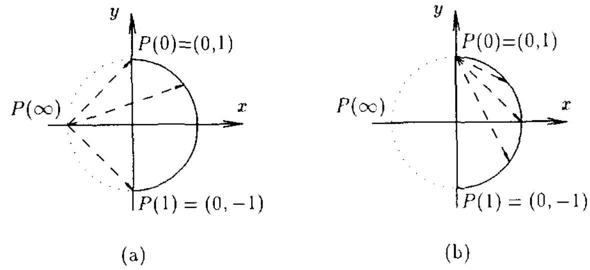


Fig. 2. The geometric interpretations of the inversion formulae (12) (Fig. (a)) and (13) (Fig. (b)).

where $B_{i,2}(t) = \binom{2}{i}t^i(1-t)^{2-i}$, $i = 0,1,2$, are second-degree Bernstein polynomials; $P_0 = (0,1)$, $P_1 = (1,0)$, and $P_2 = (0,-1)$. Writing (11) as

$$x(t) = \frac{2t - 2t^2}{1 - 2t + 2t^2}, \qquad y(t) = \frac{1 - 2t}{1 - 2t + 2t^2},$$

then using (9) and (10), we obtain two inversion formulae of $P(t)$:

$$t = \frac{x - y + 1}{2(x + 1)} \tag{12}$$

and

$$t = \frac{1 - y}{x - y + 1}. \tag{13}$$

It is easy to check that (12) is a pencil with center $P(\infty) = (-1,0)$; (13) is a pencil with center at $P(0) = (0,1)$. See Figs. 2(a) and (b).

The geometric interpretation of a rational linear inversion formula $t = \tilde{N}(x,y)/\tilde{M}(x,y)$ with center at $\tilde{A}$ is as follows. For any $(x^*,y^*) \neq \tilde{A}$, let $t^* = \tilde{N}(x^*,y^*)/\tilde{M}(x^*,y^*)$. Then $P(t^*)$ is the projection of $(x^*,y^*)$ from $\tilde{A}$ onto the curve $P(t)$. Now consider the following problem in application: Given a point $P^*$ near the curve $U(\mathcal{R}) : P(t)$, find a point $P$ on $U(\mathcal{R})$ that is closest to $P^*$. This problem arises naturally in practice. Due to accumulated computational errors resulting from the limited precision of floating point representation, it is almost impossible to give a point smack on a curve as intended to be. If one is satisfied with an approximate solution, inversion formulae can be used to solve this problem as follows. First calculate $t^* = f(x^*,y^*)$, then $P(t^*)$ can be accepted as a point on the curve that is quite close to $P^*$, though, in general, $P(t^*)$ is not the point that is closest to $P^*$ from the curve.

Consider the rational linear curve defined by (2). It is easily seen that if the center of its inversion formula is chosen to be the point with homogeneous coordinates $(b_1c_0 - b_0c_1, a_0c_1 - a_1c_0, 0)$, then the point $P(t^*)$ is the closest point to $P^*$ from the straight line $P(t)$, for the center is a point at infinity which is on all straight lines perpendicular to the line $P(t)$. But in the case of the rational quadratic curve the point $P(t^*)$ is, in general, not the closest point to $P^*$ from the curve, no matter which rational linear inversion formula is employed; however, we can still tell that the inversion formula (12) is better than (13), for in (13) if the point $P^*$ is given near the point $(0,1)$, $P(t^*)$

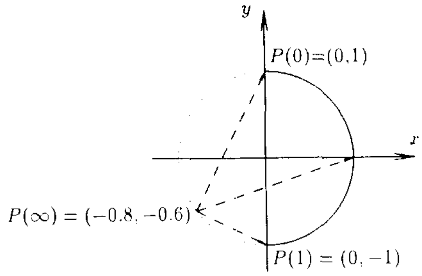


Fig. 3. The geometric interpretation of the inversion formula (13).

may be very far from $P^*$ relatively. See Fig. 2(b). In general, when the rational Bézier representation is used, (9) gives a better inversion formula than (10) because we are only interested in the curve segment defined over [0,1], and $P(\infty)$, the center of (9), is outside the Bézier segment.

Another observation is that, even when (9) is used, the center of (9) still depends on the parameterization $P(t)$, while the shape of the Bézier segment is parameterization independent. So there arises the question of what is the most appropriate parameterization of $U$ when (9) is used as an inversion formula. We claim that, in general, the standard parameterization proposed in (Patterson, 1986), i.e. setting the weights associated with the two endpoints to one, is a satisfactory choice. The reason is as follows. Suppose that $P_0P_1P_2$ is the control polygon of a rational quadratic Bézier curve $P(t)$. It is known that $P_1$, $P(1/2)$ and $P(\infty)$ are collinear (Lee, 1987). If the standard parameterization is adopted, $P(1/2)$ is the intersection of the line through $P_1$ and $M = (P_0 + P_2)/2$ with the curve $U(\mathcal{R})$. Therefore $P(\infty)$ lies on the straight line $\overline{P_1M}$. In the case in which the shape of $\Delta P_0P_1P_2$ does not differ very much from an isosceles triangle, i.e. $|P_0P_1|$ and $|P_1P_2|$ do not differ much, $P(\infty)$ is at a satisfactory position. The effect of reparameterization on the inversion formula derived from (9) is illustrated in the following example.

**Example 2.** Still consider the rational quadratic Bézier curve (11), which is a standard parameterization. Through a parameter transformation

$$t = \frac{2\hat{t}}{1+\hat{t}},$$

we obtain another parameterization of the semicircle $x^2 + y^2 = 1,\ x \geqslant 0$,

$$P(\hat{t}) = \frac{\frac{1}{4}P_0B_{0,2}(\hat{t}) + \frac{1}{2}P_1B_{1,2}(\hat{t}) + P_2B_{2,2}(\hat{t})}{\frac{1}{4}B_{0,2}(\hat{t}) + B_{2,2}(\hat{t})}, \tag{14}$$

or

$$x(\hat{t}) = \frac{4\hat{t} - 4\hat{t}^2}{1 - 2\hat{t} + 5\hat{t}^2}, \qquad y(\hat{t}) = \frac{1 - 2\hat{t} - 3\hat{t}^2}{1 - 2\hat{t} + 5\hat{t}^2},$$

which is not a standard parameterization. Using (9) we obtain the inversion formula of (14)

$$\hat{t} = \frac{2x - y + 1}{4x + 3y + 5}, \tag{15}$$

which has its center at $P(\hat{t})|_{\hat{t}=\infty} = (-\frac{4}{5}, -\frac{3}{5})$. See Fig. 3. Compared to Fig. 2(a), it is easy to tell that (12) is a better inversion formula than (15).

From (9) or (10), and the remark after the proof of Theorem 2, an inversion formula with any center $\tilde{A}$ can be computed. This may be useful if it is known that a particular $\tilde{A}$ yields a better inversion formula than one with center at $P(\infty)$ or $P(0)$.

## 4. Interpretation for rational cubic curves

The rational cubic curve is the most commonly used curve representation in CAGD. Because of its importance in application and the simplicity of its inversion formulae, this section is devoted to the discussion of rational cubic curves. We start by reviewing briefly the Bezout resultant technique for rational cubic curves. A detailed discussion is contained in (Goldman et al., 1984).

Let $P(t)$ be a faithful rational cubic plane curve given by

$$x(t) = \frac{a_3t^3 + a_2t^2 + a_1t + a_0}{d_3t^3 + d_2t^2 + d_1t + d_0}, \qquad y(t) = \frac{b_3t^3 + b_2t^2 + b_1t + b_0}{d_3t^3 + d_2t^2 + d_1t + d_0},$$

which can be rewritten as

$$\begin{aligned} &(a_3 - d_3x)t^3 + (a_2 - d_2x)t^2 + (a_1 - d_1x)t + a_1 - d_1x = 0, \\ &(b_3 - d_3y)t^3 + (b_2 - d_2y)t^2 + (b_1 - d_1y)t + b_1 - d_1y = 0. \end{aligned} \tag{16}$$

Its Bezout resultant is

$$\begin{aligned} R(x,y) &= |R_{i,j}(x,y)|_{3\times 3} \\ &= \begin{vmatrix} V_{3,2}(x,y) & V_{3,1}(x,y) & V_{3,0}(x,y) \\ V_{3,1}(x,y) & V_{3,0}(x,y) + V_{2,1}(x,y) & V_{2,0}(x,y) \\ V_{3,0}(x,y) & V_{2,0}(x,y) & V_{1,0}(x,y) \end{vmatrix}, \end{aligned} \tag{17}$$

where

$$V_{r,s}(x,y) = \begin{vmatrix} a_r - d_rx & a_s - d_sx \\ b_r - d_ry & b_s - d_sy \end{vmatrix}, \quad r,s = 0,1,2,3, \ r \neq s.$$

Then $R(x,y) = 0$ is the implicit equation of curve $P(t)$. If $(x,y)$ is given on the curve, the corresponding $t$ satisfies the system

$$R_{i,1}t^2 + R_{i,2}t + R_{i,3} = 0, \quad i = 1,2.$$

It is shown in (Goldman et al., 1984) that a rational cubic curve has a rational linear inversion formula

$$t = \frac{N(x,y)}{M(x,y)}, \tag{18}$$

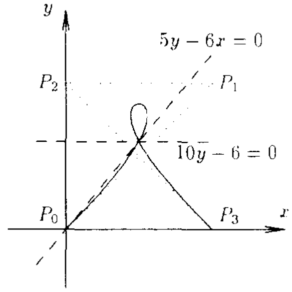


Fig. 4. Geometric interpretation of the inversion formula for the rational cubic curve given in Ex. 3, with $P_0 = (0,0)$, $P_1 = (1,1)$, $P_2 = (0,1)$, $P_3 = (1,0)$, $w_0 = w_3 = 1$, and $w_1 = w_2 = 2$. The double point is $(1/2, 3/5)$.

where $N(x,y)$ and $M(x,y) \in \mathcal{R}[x,y]$ are linear in $x$ and $y$. (It is also possible for a rational cubic curve to have a rational quadratic inversion formula.) Eq. (18) represents a pencil with center at the intersection of $N(x,y) = 0$ and $M(x,y) = 0$. Thus the geometric interpretation of an inversion formula of a rational cubic curve is a projection from some fixed point. This situation is quite similar to that of rational quadratic curves. The major difference between them is that the center of rational linear inversion formulae for a cubic curve can not be altered. We state this as the following theorem.

**Theorem 6.** *Let $U : P(t)$ be a rational cubic curve. Any rational linear inversion formula of $P(t)$ has its center $A$ at the double point of $U$. Furthermore, up to a constant for the coefficients, every rational linear inversion formula of $U$ has the same expression.*

**Proof.** Let a rational linear inversion formula be given by (18). Let $A$ be its center. Suppose that $A$ is not the singular point of the curve $U(\mathcal{C})$. Then by Lemma 1 there exists a line through $A$ that intersects $U(\mathcal{C})$ at two other distinct points, since $U(\mathcal{C})$ is an irreducible cubic. That is, there exist two distinct points on $U(\mathcal{C})$ having the same parameter value $t \in \mathcal{C}$. This contradicts the assumption that $P(t)$ is a parameterization of $U(\mathcal{C})$. Hence $A$ must be the singular point of $U$.

The second part of the theorem follows from the first part and Theorem 4. □

**Example 3.** Consider the rational cubic plane Bézier curve $P(t)$ defined by

$$P(t) = (x(t), y(t)) = \frac{\sum_{i=0}^{3} w_i P_i B_{i,3}(t)}{\sum_{i=0}^{3} w_i B_{i,3}(t)},$$

where the $B_{i,3}(t)$'s are third degree Bernstein polynomials; $P_0 = (0,0)$, $P_1 = (1,1)$, $P_2 = (0,1)$, $P_3 = (1,0)$; $w_0 = w_3 = 1$, $w_1 = w_2 = 2$. See Fig. 4. $P(t)$ can be written as

$$x(t) = \frac{7t^3 - 12t^2 + 6t}{-3t^2 + 3t + 1}, \qquad y(t) = \frac{-6t^2 + 6t}{-3t^2 + 3t + 1}.$$

Then we have

$$7t^3 + (-12+3x)t^2 + (6-3x)t - x = 0,$$
$$(-6+3y)t^2 + (6-3y)t - y = 0.$$

The Bezout resultant of the above system is

$$R = \begin{vmatrix} 7(-6+3y) & 7(6-3y) & -7y \\ 7(6-3y) & 11y-36 & 6(2y-x) \\ -7y & 6(2y-x) & 6(x-y) \end{vmatrix}.$$

The implicit equation of $P(t)$ is $R = 0$ or

$$F(x,y) \equiv 1225y^3 - 756x^2y - 2520y^2 + 756xy + 1512x^2 + 1512y - 1512x = 0.$$

Adding the first row of the above matrix to the second, we obtain

$$R = \begin{vmatrix} 7(-6+3y) & 7(6-3y) & -7y \\ 0 & 6-10y & 5y-6x \\ -7y & 6(2y-x) & 6(x-y) \end{vmatrix}.$$

Therefore $t$ satisfies

$$7(-6+3y)t^2 + 7(6-3y)t - 7y = 0,$$
$$(6-10y)t + 5y - 6x = 0.$$

Solving for $t$, we obtain an inversion formula

$$t = \frac{-\begin{vmatrix} 7(-6+3y) & -7y \\ 0 & 5y-6x \end{vmatrix}}{\begin{vmatrix} 7(-6+3y) & 7(6-3y) \\ 0 & (6-10y) \end{vmatrix}} = \frac{5y-6x}{10y-6}.$$

By Theorem 6 this inversion formula is unique with respect to $P(t)$ and the intersection of $5y - 6x = 0$ and $10y - 6 = 0$ is the double point of $P(t)$, which is $(1/2, 3/5)$. See Fig. 4.

It is also possible to obtain a rational quadratic inversion formula for the above curve. Solving for $t$ from the first and third rows of the Bezout matrix yields

$$t = \frac{y(25y-36) - x(18y-36)}{3(y-2)(5y-6x)} \equiv \frac{N_2(x,y)}{M_2(x,y)}.$$

$M_2(x,y) = 0$ and $N_2(x,y) = 0$ are two conics, with the first being degenerate, which intersect at $(0,0) = P(0)$, $(1/2, 3/5)$, and $(1,0,0)$, which are the homogeneous coordinates of $P(\infty)$; and $P(\infty)$ is a double intersection. Notice that the singular point $(1/2, 3/5)$ is still one of the intersections and all intersections are on the curve $P(t)$.

Since (18) represents a pencil that always has its center at the double point of the cubic $U$, it can be used to find the singular point of $U$ without converting the parametric form of the curve into the implicit form. All we have to do is find the inversion formula of the form (18), and then find the intersection of $M(x,y)=0$ and $N(x,y)=0$. Note that what we obtain in this way are only the coordinates of the singular point. If the corresponding parameter value(s) is wanted, the inversion formula (18) cannot be used directly, because it is indefinite at the singular point. In this case, by the properties of the Bezout resultant, besides $R=0$, the first two rows of the Bezout resultant (17) are linearly dependent, and $R_{1,1}\neq 0$. So the parameters of the singular point $(x_0,y_0)$ are the roots of the quadratic equation

$$V_{3,2}(x_0,y_0)t^2+V_{3,1}(x_0,y_0)t+V_{3,0}(x_0,y_0)=0.$$

By this way it is found that the parameter values of the singular point (1/2,3/5) in Example 3 are $(7\pm\sqrt{21})/14$.

## 5. Some general results

In this section we prove some general results for inversion formulae for the $n$th degree rational plane curve $P(t)$, $n\geqslant 3$. Let the irreducible implicit equation of $P(t)$ be $F(x,y)=0$. It is known that the inversion formula for a faithful $n$-th degree rational curve can be written as

$$t=\frac{N(x,y)}{M(x,y)},\tag{19}$$

where $N(x,y)$ and $M(x,y)$ are polynomials in $x$, $y$ of degree at most $n-2$, and they are relatively prime (Goldman et al., 1984). Note that the resultant approach mentioned in the last section is not the only way to obtain an inversion formula. If the procedure of parameterizing a rational algebraic curve is known, an inversion formula can be derived directly (Walker, 1950). The envelope of the family of curves $H(x,y;t)\equiv N(x,y)-tM(x,y)$ is the set $B$ which is determined by the system

$$H(x,y;t)=0,\qquad \partial H(x,y;t)/\partial t=0,$$

i.e. $M(x,y)=0$ and $N(x,y)=0$. Since $M(x,y)=0$ and $N(x,y)=0$ are assumed to have no common component, $B$ is a set of isolated points, which are called the *base points* of the linear system $H(x,y;t)=0$. Note that $B$ does not depend on $t$. $H(x,y;t)$ is sometimes abbreviated as $H(t)$.

If an intersection point of $F(x,y)=0$ and $H(x,y;t)=0$ is a base point of the system $H(x,y;t)=0$, then it is called a *trivial intersection*, otherwise a *variable intersection*. Since $P(t)$ is a parameterization of $F(x,y)=0$, there is only one variable intersection of $F=0$ and $H(t)=0$, which is $P(t)$. In fact, for some values of $t$, $P(t)$ can be a base point; the essential difference between the two kinds of intersections is that the former is not dependent on $t$, while the latter is.

From the discussion of the last section, two questions regarding the linear system $H(x,y;t)=0$ arise naturally: Are all base points of $H(x,y;t)=0$ on $F(x,y)=0$? Do the base points of $H(x,y;t)=0$ include all singular points of $F(x,y)=0$? The answer to the first question is, in general, negative, as illustrated in the case of rational linear curves; nontrivial examples can be given for rational curves of higher degree by using Bezout's Theorem and counting the number of intersections of $M(x,y) = 0$, $N(x,y) = 0$ and $F(x,y) = 0$, $H(x,y;t) = 0$. Nevertheless, we will see later on that it is true for a special class of curves and inversion formulae. As for the second question Goldman et al. (1984) show that all ordinary singular points of $U$ are base points of $H(x,y;t) = 0$. The main result in this section is a generalization of this result to include nonordinary singular points as well. Nonordinary singular points are often encountered, the cusp of a cubic providing a familiar example.

**Theorem 7.** *All the singular points of* $U(\mathcal{C}) : P(t)$ *are intersection points of* $M(x,y) = 0$ *and* $N(x,y) = 0$.

Before giving the proof we need some conventions on notation. For the concepts involved the reader is referred to (Walker, 1950). The order of curve $F(x,y) = 0$ is denoted by $\mathrm{Ord}(F)$, which is the degree of $F(x,y)$. The multiplicity of a point $P = (a,b)$ on $F(x,y) = 0$ is denoted by

$$M_P \equiv \min \left\{ i \;\middle|\; \exists j,\ 0 \leqslant j \leqslant i,\ \left.\frac{\partial^i F(x,y)}{\partial^j x \partial^{i-j} y}\right|_{(a,b)} \neq 0,\ 0 \leqslant i \leqslant \mathrm{Ord}(F) \right\}.$$

A place $v(P)$ of $U(\mathcal{C})$ with center at point $P$ is an equivalence class of irreducible power series parameterizations $(\bar{x}(t), \bar{y}(t))$ of $U(\mathcal{C})$ with centers at $P$. Here the center $(\bar{x}(0), \bar{y}(0))$ of a place is to be distinguished from the center of a rational linear inversion formula defined previously. $O_{v(P)}(F)$ denotes the order of $F(x,y)$ at the place $v(P)$, which is the lowest exponent of the power series $F(\bar{x}(t), \bar{y}(t))$, where $(\bar{x}(t), \bar{y}(t))$ is a representative of the equivalence class $v(P)$. Of course $O_{v(P)}(F)$ is independent of the choice of the representative $(\bar{x}(t), \bar{y}(t))$. For two curves $F(x,y) = 0$ and $G(x,y) = 0$, with no common components, it can be shown that $\sum O_{v(P)}(G) = \sum O_{w(P)}(F)$, where the first sum is taken over all places of $F$ with centers at a point $P$, and the second sum is taken over all places of $G$ with centers at the same point $P$; therefore $I_P(F,G) := \sum O_{w(P)}(F) = \sum O_{v(P)}(G)$ is well defined and is called the *number of intersections* of $F(x,y) = 0$ and $G(x,y) = 0$ at point $P$. Let $I(F,G) = \sum I_P(F,G)$, with the sum taken over all points common to $F(x,y) = 0$ and $G(x,y) = 0$; $I(F,G)$ is called the number of intersections of $F = 0$ and $G = 0$. We also need the following three lemmas from (Walker, 1950), which are given here without proof.

**Lemma 2** (Bezout's Theorem). *When all intersection points of* $F(x,y) = 0$ *and* $G(x,y) = 0$ *are finite points,*

$$I(F,G) = \mathrm{Ord}(F)\mathrm{Ord}(G).$$

**Lemma 3.**

$$I_P(F,G) \geqslant M_P(F) M_P(G),$$

*where the equality holds if and only if* $F = 0$ *and* $G = 0$ *have no common tangent at* $P$.

Let $R_y(F,G)$ be the resultant of $F(x,y)$ and $G(x,y)$ with respect to $y$. Let $P$ be an intersection of $F=0$ and $G=0$, and let the abscissa of $P$ be $x_P$.

**Lemma 4.** *If all other points common to $F=0$ and $G=0$, besides $P$, do not have the abscissa $x_P$, then $I_P(F,G)$ equals the multiplicity of $x_P$ as a root of $R_y(F,G)$.*

**Proof of Theorem 7.** Let the irreducible equation of $U(\mathcal{C}) : P(t)$ be $F(x,y)=0$. Let $T$ be the set of all trivial intersections of $F(x,y)=0$ and $H(x,y;t)=0$. Then $T \subset B$, the set of all base points of $H(x,y;t)=0$. The only nontrivial, or variable, intersection between $F(x,y)=0$ and $H(x,y;t)=0$ is $P(t)$, and $T \cup \{P(t)\}$ is the set of all intersections of $F=0$ and $H(t)=0$. We assume that all singular points of $F(x,y)=0$ and points in $T$ are finite points and have different abscissae; this can be achieved by appropriate choices of the line at infinity and an affine representation of the plane. Note that the property we want to prove is not affected by such choices.

We need to show that all the singular points of $F(x,y)=0$ are trivial intersections with $H(x,y;t)=0$. Suppose $P_0=(x_0,y_0)$ is a singular point of $U(\mathcal{C})$, i.e. $M_{P_0}(F)>1$, but $P_0 \notin T$. We will proceed to obtain a contradiction.

From the above supposition it follows that $M(x_0,y_0)$ and $N(x_0,y_0)$ are not both zero. Thus a definite and unique parameter value $t_0$ can be obtained from (19) such that $H(x_0,y_0;t_0)=0$ and $P_0=P(t_0)$. By reparameterization, it can be assumed that $t_0$ is finite.

The concept of parameterizations of a place on an algebraic curve enables us to talk about points on the curve which lie in the neighborhood of a given point on the curve. Evidently, $\tilde{P}(t) := P(t+t_0)$ is a parameterization of a place with center at $P_0$ since $\tilde{P}(0)=P(t_0)$, $t$ being in a neighborhood of 0 in $\mathcal{C}$. We claim that we can choose $t_1=t_0+\delta$, with $|\delta|$ being so small that the following conditions are observed for $P_1 \equiv P(t_1)$:

(i) $M_{P_1}(F)=1$, i.e. $P_1=\tilde{P}(\delta)$ is a simple point of $F(x,y)=0$, and $P_1 \notin T$. This is possible because $F(x,y)=0$ has only a finite number (at most $\frac{1}{2}(n-1)(n-2)$) of singular points and $|T|$ is finite. Hence

$$M_{P_1}(F) < M_{P_0}(F). \tag{20}$$

(ii) For any $P=(x_P,y_P)\in T$

$$I_P(F,H(t_1)) \leqslant I_P(F,H(t_0)).$$

This is because, by Lemma 4, $I_P(F,H(t))$ is the multiplicity of $x_P$ as a root of $R_y(F,H(t))$, which is a continuous function of $t$. So a sufficiently small change in $t$ will not increase, but will probably decrease $I_P(F,H(t))$. Consequently

$$\sum_{P\in T} I_P(F,H(t_1)) \leqslant \sum_{P\in T} I_P(F,H(t_0)). \tag{21}$$

(iii) $$M_{P_1}(H(t_1)) \leqslant M_{P_0}(H(t_0)). \tag{22}$$

This is because

$$M_{P(t)} = \min\left\{ i \;\middle|\; \exists j,\ 0 \leqslant j \leqslant i,\ \left.\frac{\partial^i F(x,y)}{\partial^j x \partial^{i-j} y}\right|_{P(t)} \neq 0,\ 0 \leqslant i \leqslant \mathrm{Ord}(F) \right\},$$

and each partial derivative of $F$ evaluated at $P(t)$ is a continuous function of $t$. So a small change in $t$ will not increase, but will probably decrease $M_{P(t)}(H(t))$.

(iv) $$\mathrm{Ord}(H(t_1)) \geqslant \mathrm{Ord}(H(t_0)), \tag{23}$$

for the coefficients of the highest degree terms of $H(x,y;t) = 0$ are continuous functions of $t$.

(v) $$I_{P_1}(F, H(t_1)) = M_{P_1}(F) M_{P_1}(H(t_1)). \tag{24}$$

By Lemma 3 this holds if $F(x,y) = 0$ and $H(x,y;t_1) = 0$ have no common tangent at $P_1$. If for any $t$ in a neighborhood of $t_0$ these two curves always have common tangents at $P(t)$, then by definition, a nontrivial subcurve of $F(x,y) = 0$ is in the envelope of the family $H(x,y;t) = 0$. But this contradicts the fact that the envelope of $H(x,y;t) = 0$ is $B$, the set of all its base points. So we can select $t_1$ in a neighborhood of $t_0$ such that (24) holds.

Now we show that the above conditions lead to a contradiction. Note that $T \cup \{P(t_1)\}$ is the set of all intersections of $F(x,y) = 0$ and $H(x,y;t_1) = 0$. Let $\mathrm{Ord}(H(t_1)) = m$. By Lemma 2,

$$\begin{aligned}
nm &= \mathrm{Ord}(F)\mathrm{Ord}(H(t_1)) \\
&= \sum_{P\in T} I_P(F, H(t_1)) + I_{P_1}(F, H(t_1)) \\
&= \sum_{P\in T} I_P(F, H(t_1)) + M_{P_1}(F) M_{P_1}(H(t_1)) && \text{(by (24))} \\
&< \sum_{P\in T} I_P(F, H(t_0)) + M_{P_0}(F) M_{P_0}(H(t_0)) && \text{(by (21), (20), and (22))} \\
&\leqslant \sum_{p\in T} I_P(F, H(t_0)) + I_{P_0}(F, H(t_0)) && \text{(by Lemma 3)} \\
&= \mathrm{Ord}(F)\mathrm{Ord}(H(t_0)) \leqslant nm && \text{(by Lemma 2 and (23)).}
\end{aligned}$$

This contradiction implies that $P_0 \in T$. □

Theorem 7 encompasses the case of rational cubic curves. But since for rational cubic curves the inversion formula is known to be expressible as a rational linear function, it is possible to give a much simpler proof. The theorem is also valid for inversion formulae of any degree $i$, not necessarily that $i \leqslant n-2$. An implication of Theorem 7 is that when a linear system is to be determined to give a rational parameterization of a rational algebraic plane curve $F(x,y) = 0$, the system must be required to pass through all the singular points of $F(x,y) = 0$, as the approaches described in (Abhyankar et al., 1988) and (Walker, 1950).

The geometric interpretation of a general inversion formula $t = N(x,y)/M(x,y)$ can be given as follows. Let $P^* = (x^*, y^*) \notin B$ be a point not on $U : P(t)$. There is a unique

$t^* = N(x^*, y^*)/M(x^*, y^*)$ such that $H(x, y; t^*) = 0$ passes through $P^*$. Then $P(t^*)$, the point on $U$ corresponding to $P^*$, is the only variable intersection of $F(x, y) = 0$ and $H(x, y; t^*) = 0$. Theorem 7 provides an important property of the linear system $H(x, y; t) = 0$, which plays a crucial role in the geometric interpretation of an inversion formula. As an analogue to the case of rational linear inversion formulae we can think of the above interpretation of higher degree inversion formulae as a generalized projection from the plane into the curve $P(t)$ in question.

The following discussion is to improve the understanding of rational quadratic inversion formulae which come about naturally in the case of rational quartic curves. A rational quartic plane curve has either a triple point or three distinct double points. In the former case there always exists a rational linear inversion formula for $P(t)$. In the latter case the minimum degree inversion formulae are quadratic. In this case we can show that all base points of $H(x, y; t) = 0$ are on $P(t)$.

**Theorem 8.** *Let $P(t)$ be a rational quartic plane curve with three distinct double points, and $t = N(x, y)/M(x, y)$ be a rational quadratic inversion formula for $P(t)$. Then all base points of $H(x, y; t) \equiv N(x, y) - tM(x, y) = 0$ are on $P(t)$.*

**Proof.** Let the implicit equation of $P(t)$ be $F(x, y) = 0$. By Bezout's Theorem $H(x, y; t) = 0$ has at most four distinct base points, as intersections of two conics. If a base point is an intersection of multiplicity one between $M(x, y) = 0$ and $N(x, y) = 0$, then it is called a *simple base point*. It is obvious that there are exactly four distinct base points of $H(x, y; t) = 0$ if and only if all the base points are simple. By Theorem 7, $H(x, y; t) = 0$ has a base point at each of the three double points of $F(x, y) = 0$. Assume that $H(x, y; t) = 0$ has four distinct base points; for otherwise there is nothing to prove. Therefore all the four base points must be simple. Suppose that a base point is not on $F = 0$. Because at each double point, except for a finite number of values of $t$, the number of intersections of $F = 0$ and $H(t) = 0$ is two, there exists $t$ such that, including the variable intersection $P(t)$, the total number of intersections of $F = 0$ and $H(t) = 0$ is $3 \times 2 + 1 = 7$, contradicting Bezout's Theorem which asserts that the number of intersections of a quartic curve and a conic is eight. Hence every base point of $H(x, y; t) = 0$ is on $F(x, y) = 0$. □

Theorem 8 and the geometric interpretation of rational quadratic inversion formulae are illustrated in the following example.

**Example 4.** Let a rational quartic curve be $P(t) = (x(t), y(t))$,

$$x(t) = \frac{-t^4 - 2t^3 - 2t + 1}{t^4 - 4t^3 - 6t^2 - 4t + 1}, \qquad y(t) = \frac{t^4 - 2t^3 - 2t - 1}{t^4 - 4t^3 - 6t^2 - 4t + 1}.$$

Its implicit equation is

$$F(x, y) \equiv x^4 + 2x^3y + 2xy^3 + y^4 - 2x^3 - 2x^2y - 2xy^2 - 2y^3 + x^2 + y^2 = 0,$$

which has three distinct double points $(0,0) = P(\pm i)$, $(1,0) = P(1 \pm \sqrt{2})$, and $(0,1) = P(-1 \pm \sqrt{2})$. The point $(0,0)$ is an isolated real point. See Fig. 5. An

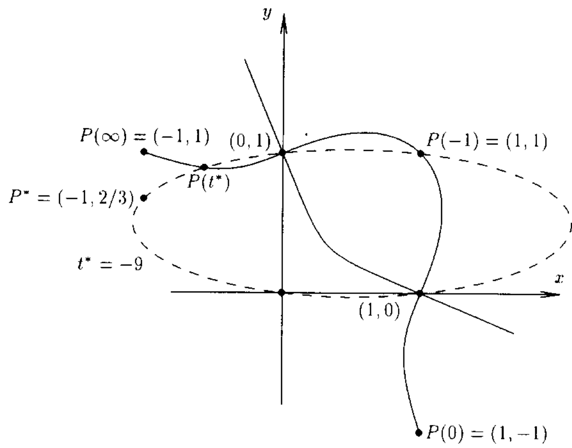


Fig. 5. The rational quartic curve $P(t)$, given in Example 4, consists of the two solid curve segments in the figure. There are three double points: $P(-1 \pm \sqrt{2}) = (0,1)$, $P(1 \pm \sqrt{2}) = (1,0)$, and $P(\pm i) = (0,0)$, which is an isolated real point. The dashed ellipse is the curve $H(x,y;t^*) = 0$, where $t^* = -9$, determined by the point $P^* = (-1,2/3)$, whose corresponding point $P(t^*) \approx (-0.56314, 0.89012)$ on $P(t)$ is the variable intersection of $P(t)$ and $H(x,y;t^*) = 0$.

inversion formula for the curve is

$$t = \frac{x(x-1)}{y(y-1)}.$$

Therefore $H(x,y;t) = x(x-1) - ty(y-1)$, which has four distinct base points, including the three double points and a simple point $(1,1)$ of the curve $F(x,y) = 0$.

For a point $P^* = (-1,2/3)$ not on $P(t)$, the corresponding parameter value is found to be $t^* = -9$; therefore $H(x,y;t^*) = x(x-1) + 9y(y-1)$. The variable intersection of $H(x,y;t^*) = 0$ and $P(t)$ gives the corresponding point $P(t^*) \approx (-0.56314, 0.89012)$ of $P^*$ on $P(t)$, i.e. $P(t^*)$ is obtained from a conic projection of $P^*$ instead of a straight line projection. See Fig. 5.

## 6. Summary

The geometric interpretation of inversion formulae for rational plane curves is studied. When the rational curve is of degree $k$, $k = 1, 2$ or 3, the inversion formula can be written as rational linear function $t = N(x,y)/M(x,y)$ in $x$ and $y$. For a point $P^* = (x^*, y^*)$ not on $P(t)$, the corresponding point $P(t^*)$ on the curve, where $t^* = N(x^*,y^*)/M(x^*,y^*)$, is the projection of $P^*$ from some point $A$. When $k = 1$, $A$ can be any point in the plane but not on curve $P(t)$; when $k = 2$, $A$ can be any point on the curve $P(t)$; when $k = 3$, $A$ must be the only double point of the curve. The geometric interpretation of an inversion formula $t = N(x,y)/M(x,y)$ for a general rational plane curve $P(t)$ is a generalized projection from the plane into the curve $P(t)$. This generalized projection is determined by the linear system $H(x,y;t) \equiv N(x,y) - tM(x,y) = 0$. It is shown that all the singular points of $P(t)$ are base points of $H(x, y; t) = 0$. The applications of these results are discussed through some examples.

There are still some open problems in the study of inversion formulae for rational plane curves; for instance, how to find a minimum degree inversion formula for a general rational plane curve, and when it is not unique how to characterize all minimum degree inversion formulae. And further research is needed on the inversion formulae for rational space curves and rational surfaces.

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